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Scalar radiation emitted from a source rotating around a black hole

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Abstract

We analyze the scalar radiation emitted from a source rotating around a Schwarzschild black hole using the framework of quantum field theory at the tree level. We show that for relativistic circular orbits the emitted power is about 20% to 30% smaller than what would be obtained in Minkowski spacetime. We also show that most of the emitted energy escapes to infinity. Our formalism can readily be adapted to investigate similar processes.

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I. INTRODUCTION

Observational confirmation of the existence of black holes is one of the most important challenges in astrophysics. Recently a number of compact objects in X-ray binary systems have been identified as black holes since a careful analysis has shown that their masses are far beyond any limit accepted for dead stars in general relativity [1]. There also exists indirect evidence of the presence of supermassive black holes in the centre of some galaxies [2]. Nevertheless, unambiguous confirmation of the existence of black holes would require the observation of effects due to the event horizon itself. This is expected to be achieved through precise measurements of the electromagnetic radiation emitted from black hole accretion disks (see e.g. [3,4]), and from the gravitational radiation emitted from companion stars orbiting black holes (see e.g. [5]). Because radiation from sources orbiting black holes plays such a crucial role in modern astrophysics and because increasingly precise measurements are leading to the observation of relativistic effects occurring in the vicinity of the horizon [6], investigation of how radiation-emission processes are modified by the nontrivial curvature and topology of the black-hole spacetime is particularly important.

In this paper we analyze analytically and numerically the *scalar* radiation emitted by a source rotating around a black hole using the framework of quantum field theory at the tree level, and compare the results with those obtained in Newtonian gravity and in a theory associated with one-graviton exchange in flat spacetime. We show that these results coincide asymptotically, as expected, but considerably differ close to the last stable circular orbit. We also calculate the amount of emitted energy which is not absorbed by the black hole. The paper is organized as follows. In section II we present the framework of quantum field theory in which we will work. In section III we obtain analytical and numerical results for the emitted power. In section IV we establish a connection between our quantum field theory approach and classical field theory. In section V we compare our curved spacetime calculation with flat spacetime calculations by considering (i) Newtonian gravity and (ii) the one-graviton exchange theory. In Section VI we calculate analytically and numerically the amount of radiation which is not absorbed by the hole. Finally in section VII we make some remarks on our results. We use natural units $c = \hbar = G = 1$ and signature $(+ - - -)$ throughout this paper.

II. GENERAL FRAMEWORK

A nonrotating black hole with mass M is described by the Schwarzschild line element

$$ds^2 = f(r)dt^2 - f(r)^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2 \quad , \tag{1}$$

where $f(r) = 1 - 2M/r$. Let us consider a circularly moving scalar source at $r = R_S$ with constant angular velocity $\Omega > 0$ (as measured by asymptotic static observers) on the plane $\theta = \pi/2$ described by

$$j(x^\mu) = \frac{q}{\sqrt{-g}\; u^0} \delta(r - R_S) \; \delta(\theta - \pi/2) \; \delta(\phi - \Omega t) \tag{2}$$

with $g \equiv det(g_{\mu\nu})$, where the constant q determines the magnitude of the source-field coupling. The four-velocity of this source is

$$u^\mu(\Omega, R_S) = [(f(R_S) - R_S^2\Omega^2)^{-1/2}, 0, 0, \Omega/(f(R_S) - R_S^2\Omega^2)^{1/2}] \ . \quad (3)$$

We have normalized the source j by requiring that $\int d\sigma j(x^\mu) = q$, where $d\sigma$ is the proper three-volume element orthogonal to u^μ. Let us now minimally couple $j(x^\mu)$ to a massless scalar field $\hat{\Phi}(x^\mu)$ so that the total Lagrangian density is

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}\nabla^\mu\hat{\Phi}\nabla_\mu\hat{\Phi} + j\hat{\Phi}\right) \ . \quad (4)$$

The positive-frequency modes in spherically symmetric static spacetime can be given in the form

$$u_{\omega lm}(x^\mu) = \sqrt{\frac{\omega}{\pi}}\frac{\psi_{\omega l}(r)}{r}Y_{lm}(\theta,\phi)e^{-i\omega t} \quad (\omega > 0) \ , \quad (5)$$

where $\Box u_{\omega lm} = 0$. In the present case the functions $\psi_{\omega l}(r)$ satisfy the differential equation

$$\left[-f(r)\frac{d}{dr}\left(f(r)\frac{d}{dr}\right) + V_S(r)\right]\psi^S_{\omega l}(r) = \omega^2\psi^S_{\omega l}(r) \quad (6)$$

with the following scattering potential (see figure 1):

$$V_S(r) = (1 - 2M/r)\left[2M/r^3 + l(l+1)/r^2\right] \ . \quad (7)$$

In terms of the dimensionless tortoise coordinate $x \equiv r/2M + \ln(r/2M - 1)$, (6) can be rewritten as

$$\left[-\frac{d^2}{dx^2} + 4M^2V_S[r(x)]\right]\psi^S_{\omega l} = 4M^2\omega^2\psi^S_{\omega l} \ . \quad (8)$$

The scalar field can be expanded in terms of the complete set of positive- and negative-frequency modes as

$$\hat{\Phi}^{\rm in}(x^\mu) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_0^\infty d\omega\left[u_{\omega lm}(x^\mu)a^{\rm in}_{\omega lm} + H.c.\right] \ . \quad (9)$$

The Klein-Gordon inner product σ_{KG} is defined by

$$\sigma_{KG}(\phi,\psi) = i\int_{\Sigma_t} d\Sigma\, n^\mu(\phi^*\nabla_\mu\psi - \nabla_\mu\phi^*\cdot\psi) \ , \quad (10)$$

where n^μ is the future-pointing unit vector orthogonal to the Cauchy surface Σ_t with $t = const$. The modes $u_{\omega lm}$ are Klein-Gordon orthonormalized:

$$\sigma_{KG}(u_{\omega lm}, u_{\omega' l' m'}) = \delta(\omega - \omega')\delta_{ll'}\delta_{mm'} \ , \quad (11)$$
$$\sigma_{KG}(u^*_{\omega lm}, u_{\omega' l' m'}) = 0 \ . \quad (12)$$

Then the creation and annihilation operators, $a^{\rm in\dagger}_{\omega lm}$ and $a^{\rm in}_{\omega lm}$, satisfy the usual commutation relations $[a^{\rm in}_{\omega lm}, a^{\rm in\dagger}_{\omega' l' m'}] = \delta(\omega - \omega')\delta_{ll'}\delta_{mm'}$.

We see from (8) that close to and far away from the horizon, the modes $\overrightarrow{\psi}^{S}_{\omega l}(x)$ purely incoming from the past horizon H^-, and the modes $\overleftarrow{\psi}^{S}_{\omega l}(x)$ purely incoming from the past null infinity $\mathcal{J}^-$ can be written as [7,8]

$$\overrightarrow{\psi}^{S}_{\omega l}(x) \approx \begin{cases} A_{\omega l}(e^{2iM\omega x} + \overrightarrow{\mathcal{R}}_{\omega l}\, e^{-2iM\omega x}) \quad (r \gtrsim 2M)\ , \\ 2i^{l+1} A_{\omega l}\, \overrightarrow{\mathcal{T}}_{\omega l}\, M\omega x\, h_l^{(1)}(2M\omega x) \quad (r \gg 2M)\ , \end{cases} \tag{13}$$

and

$$\overleftarrow{\psi}^{S}_{\omega l}(x) \approx \begin{cases} A_{\omega l}\, \overleftarrow{\mathcal{T}}_{\omega l}\, e^{-2iM\omega x} \quad (r \gtrsim 2M)\ , \\ 2A_{\omega l} M\omega x[(-i)^{l+1} h_l^{(2)}(2M\omega x) + i^{l+1}\, \overleftarrow{\mathcal{R}}_{\omega l}\, h_l^{(1)}(2M\omega x)] \quad (r \gg 2M)\ , \end{cases} \tag{14}$$

where $h_l^{(i)}$ $(i = 1, 2)$ are spherical Bessel functions of the third kind [9] and the overall normalization constant is determined by (11) as $A_{\omega l} = (2\omega)^{-1}$.

The emitted power is given by

$$W^{em}_{lm} = \int_0^{+\infty} d\omega\, \omega\, |\mathcal{A}^{em}_{\omega lm}|^2/T\ , \tag{15}$$

where

$$\mathcal{A}^{em}_{\omega lm} = \langle \omega lm|\, i \int d^4x \sqrt{-g}\, j(x^\mu)\hat{\Phi}(x^\mu)\, |0\rangle = i \int d^4x \sqrt{-g}\, j(x^\mu) u^*_{\omega lm}(x^\mu) \tag{16}$$

is the emission amplitude at the tree level, and $T = 2\pi\delta(0)$ is the total time measured by asymptotic observers [10]. We have chosen the initial state $|0\rangle$ to be the Boulware vacuum, i.e. $a^{\rm in}_{\omega lm}|0\rangle = 0$. If we had chosen the Unruh or Hartle-Hawking vacuum [11], then (15) would be associated with the *net* radiation emitted from the source since the absorption and stimulated emission rates (which are induced by the presence of thermal fluxes) are exactly the same. Note that for sources in constant circular motion the amplitude $\mathcal{A}^{em}_{\omega lm}$ is proportional to $\delta(\omega - m\Omega)$. Hence the frequency of emitted waves is constrained by $\omega = m\Omega$. In particular, since $\Omega > 0$, no waves with $m \leq 0$ are emitted.

III. EMITTED POWER: NUMERICAL AND ANALYTICAL RESULTS

The general solution of (8) is not easy to analyze [12] and therefore we first estimate the radiated power numerically. (Later we will compare it with the one obtained with our analytic approach.) To do so we solve (8) for the left- and right-moving radial functions $\overleftarrow{\psi}^{S}_{\omega l}(r)$ and $\overrightarrow{\psi}^{S}_{\omega l}(r)$ with asymptotic boundary conditions compatible with equations (13) and (14). The method for finding the the functions $\overleftarrow{\psi}^{S}_{\omega l}$ can be summarized as follows. (The functions $\overrightarrow{\psi}^{S}_{\omega l}$ are obtained in a similar manner.) We recall from (14) that close to the horizon $\overleftarrow{\psi}^{S}_{\omega l} \propto \exp(-2iM\omega x)$. Thus we construct the solution $\chi_{\omega l}(x)$ of (8) with fixed ω satisfying $\chi_{\omega l}(x) = \exp(-2iM\omega x)$ as the initial condition for $x < 0, |x| \gg 1$ by evolving it numerically towards large x. Far away from the horizon we have $\chi_{\omega l}(x) \approx A\exp(-2iM\omega x) + B\exp(2iM\omega x)$ [see (14) and recall

that $h_l^{(1)}(2M\omega x) \approx (-i)^{l+1}e^{2iM\omega x}/(2M\omega x)$ and $h_l^{(2)}(2M\omega x) \approx i^{l+1}e^{-2iM\omega x}/(2M\omega x)$ for $2M\omega x \gg 1$]. The constants A and B (with $|A|^2 - |B|^2 = 1$) are determined from the numerically obtained values of $\chi_{\omega l}(x)$ and $d\chi_{\omega l}/dx$ for $2M\omega x \gg 1$. Now, by requiring the asymptotic boundary conditions compatible with (14), we find $\overleftarrow{\psi}^S_{\omega l}(x) = \chi_{\omega l}(x)/(2\omega A)$. (In particular, $\overleftarrow{\mathcal{T}}_{\omega l} = 1/A$ and $\overleftarrow{\mathcal{R}}_{\omega l} = B/A$.)

By substituting the radial functions $\overleftarrow{\psi}^S_{\omega l}$ and $\overrightarrow{\psi}^S_{\omega l}$ in (5) to construct the right- and left-moving modes $\overrightarrow{u}_{\omega lm}$ and $\overleftarrow{u}_{\omega lm}$, respectively, and using (15) with (16), we find the corresponding radiated powers:

$$\overrightarrow{W}^{S,em}_{lm} = 2m^2\Omega^2 q^2[f(R_S) - R_S^2\Omega^2] \, |\overrightarrow{\psi}^S_{\omega_0 l}(R_S)/R_S|^2 \, |Y_{lm}(\pi/2, \Omega t)|^2 \ , \tag{17}$$

where $\omega_0 \equiv m\Omega$ and $l, m \geq 1$, and similarly for $\overleftarrow{W}^{S,em}_{lm}$. We note [13] that $Y_{lm}(\pi/2, \Omega t) = 0$ if $l + m$ is odd and

$$|Y_{lm}(\pi/2, \Omega t)|^2 = \frac{2l+1}{4\pi}\frac{(l+m-1)!!(l-m-1)!!}{(l+m)!!(l-m)!!} \tag{18}$$

if $l + m$ is even, which is of course time independent. We have defined $n!! \equiv n(n-2)\cdots 1$ if n is odd and $n!! \equiv n(n-2)\cdots 2$ if n is even and $(-1)!! \equiv 1$. Now, the condition that our source be in a stable circular geodesic implies $R_S = (M\Omega^{-2})^{1/3}$ as is well known [14]. By using this formula, $\overrightarrow{W}^{em}_{lm}$ and $\overleftarrow{W}^{em}_{lm}$ can be cast as functions of quantities measured at infinity alone, namely, Ω and M. In figure 2 we plot the total radiated power with fixed angular momentum, $W^{S,em}_{lm} = \overrightarrow{W}^{S,em}_{lm} + \overleftarrow{W}^{S,em}_{lm}$, as a function of the angular velocity Ω for different values of l and m. Note that $W^{S,em}_{lm} = 0$ for odd $l+m$ because $Y_{lm}(\pi/2, \Omega t) = 0$ in this case.

Next we consider an analytic approximation valid for low-frequency modes. Let us first recall that waves emitted from circularly moving sources obey the constraint $\omega = m\Omega$. Thus, waves with $m = 1$ (which turn out to be the most important ones for our purposes) emitted from sources in stable circular geodesic orbits ($R_S > 6M$, $\Omega = \sqrt{M/R_S^3}$) have maximum frequency $\omega^{\max} = (6\sqrt{6}\ M)^{-1} \ll \sqrt{V_S^{\max}}$, where $V_S^{\max}$ is the maximum of the scattering potential V_S. Hence waves with small angular momentum have small frequencies in comparison with $\sqrt{V_S^{\max}}$, i.e., $\omega^2/V_S^{\max} < (\omega^{\max})^2/V_S^{\max} \approx 4 \times 10^{-2} \ll 1$ (see figure 1). A similar analysis for waves with arbitrary m shows that $\omega^2/V_S^{\max} < 10^{-1}$. Therefore, we will consider the approximation where the radial functions are replaced by their leading terms for small ω (see equations (6.6) and (7.2) in [8], and [15] for a correction):

$$\overrightarrow{\psi}^S_{\omega l}(r) \approx 2rQ_l(r/M - 1) \tag{19}$$

and

$$\overleftarrow{\psi}^S_{\omega l}(r) \approx \frac{2^{2l}(l!)^3(M\omega)^l rP_l(r/M-1)}{(2l)!\,(2l+1)!} \ , \tag{20}$$

where $P_l(x)$ and $Q_l(x)$ are the Legendre functions. By substituting $\overrightarrow{\psi}^S_{\omega l}$ and $\overleftarrow{\psi}^S_{\omega l}$ (given in equations (19) and (20) respectively) in (17) and a similar expression for $\overleftarrow{W}^{em}_{lm}$, we find the corresponding radiated powers [$R_S = (M\Omega^{-2})^{1/3}$]:

$$\vec{W}{}^{S,em}_{lm} \approx 8q^2m^2\Omega^2 \left(f(R_S) - R_S^2\Omega^2\right) |Q_l(R_S/M-1)|^2 \, |Y_{lm}(\pi/2, \Omega t)|^2 \tag{21}$$

and

$$\overset{\leftarrow}{W}{}^{S,em}_{lm} \approx \frac{2^{4l+1}q^2(l!)^6 m^{2l+2}M^{2l}\Omega^{2l+2}}{[(2l)!]^2[(2l+1)!]^2} \left(f(R_S) - R_S^2\Omega^2\right) |P_l(R_S/M-1)|^2 \, |Y_{lm}(\pi/2, \Omega t)|^2 \tag{22}$$

where $l, m \geq 1$. The total radiated power with fixed angular momentum obtained in this approximation is also shown in figure 2. We see from this figure that our analytic approximation has better accuracy for small Ω as expected. [Note that $\omega \propto \Omega$ and see discussion above (19)].

It can be seen from figure 2 that the total radiated power

$$W^{S,em} = \sum_{l=1}^{\infty} \sum_{m=1}^{l} W^{S,em}_{lm} \tag{23}$$

will be dominated by waves with small l. For circular geodesic orbits far enough from the horizon we can use equations (21)-(23) to write the radiated power in a simple form:

$$W^{S,em}\Big|_{R \gg r_S} \approx q^2 M^{2/3} \Omega^{8/3}/12\pi \ . \tag{24}$$

IV. CONNECTION WITH CLASSICAL FIELD THEORY

Since our calculations are performed at the tree level, our results can be interpreted in classical field theory as follows (see, e.g., [16]). One can show that the energy of a classical field ϕ can be written as

$$E = \frac{i}{2} \sigma_{KG}(\phi, \partial_t \phi) \ , \tag{25}$$

where the Klein-Gordon inner product σ_{KG} is defined by (10). The classical field generated by a source $j(x^\mu)$ can be expressed in general as

$$\phi(x^\mu) = i \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{+\infty} d\omega \left[\mathcal{A}^{em}_{\omega lm} u_{\omega lm}(x^\mu) - \mathcal{A}^{em*}_{\omega lm} u_{\omega lm}(x^\mu)^*\right] \ , \tag{26}$$

where $\mathcal{A}^{em}_{\omega lm}$ is defined by equation (16). By substituting this in (25) and using the orthonormality of $u_{\omega lm}$ we obtain the energy as

$$E = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int_0^{+\infty} d\omega \ \omega |\mathcal{A}^{em}_{\omega lm}|^2 \ , \tag{27}$$

which agrees with (15). (See, e.g., [17] for other classical-field-theory analyses mainly developed to study gravitational wave emission.)

V. CURVED VS. FLAT SPACETIME CALCULATIONS

Now we compare the radiated power $W^{S,em}$ in Schwarzschild spacetime with that in Minkowski spacetime. We consider radiation from a source in circular motion in Minkowski spacetime due to the presence of some gravitational force. The latter should give fairly good results for the case of a source rotating around a star that is not very dense, but not for the case of a source rotating close to a black hole as will be shown.

Let us represent the scalar source in Minkowski spacetime by

$$j^M(x^\mu) = \frac{q}{R_M^2\,\gamma}\delta(r - R_M)\;\delta(\theta - \pi/2)\;\delta(\phi - \Omega t) \quad , \tag{28}$$

where $\gamma = 1/\sqrt{1 - R_M^2\Omega^2}$. We are working here with spherical coordinates defined through the Minkowski line element given by (1) with $f(r) = 1$. Note that the normalization of this source is chosen so that $\int d\sigma\, j^M(x^\mu) = q$, where $d\sigma$ is the proper three-volume element orthogonal to the world line of the source, as in the Schwarzschild case.

It is not *a priori* meaningful to compare the radiated powers from the sources in Schwarzschild and Minkowski spacetimes with the same value of r. For this reason, our results will be expressed in terms of Ω. It is meaningful to compare the radiated powers using Ω because this is a (coordinate free) quantity measured by asymptotic observers. We expand the scalar field as in equation (9) with Klein-Gordon orthonormalized positive-frequency modes given by (5), where the radial functions are given by $\psi^M_{\omega l}(r) = r j_l(\omega r)$ (see, e.g., [8]), which satisfy the differential equation

$$\left(-d^2/dr^2 + V_M\right)\psi^M_{\omega l}(r) = \omega^2 \psi^M_{\omega l}(r) \tag{29}$$

with $V_M \equiv l(l+1)/r^2$. The total radiated power calculated in Minkowski spacetime is

$$W^{M,em} = \sum_{l=1}^{\infty}\sum_{m=1}^{l} 2q^2 m^2 \Omega^2 \gamma^{-2}\,|j_l(m\Omega R_M)|^2 |Y_{lm}(\pi/2, \Omega t)|^2 \quad , \tag{30}$$

where we have used the Minkowski vacuum as the initial state. In order to compare this expression with the radiated power obtained through classical field theory, we adapt the standard derivation of Larmor's formula for electric charges (see, e.g., [18]) to the case of scalar sources:

$$W^{M,em}_{class} = q^2 a^2 / 12\pi \quad , \tag{31}$$

where a is the proper acceleration of the source with an arbitrary trajectory. For circular orbits, we have $a = \gamma^2 \Omega^2 R_M$. The equality $W^{M,em} = W^{M,em}_{class}$ follows from the following formula:

$$\sum_{l=1}^{\infty}\sum_{m=1}^{l} m^2 [j_l(mz)]^2 |Y_{lm}(\pi/2, \phi)|^2 = \frac{1}{24\pi}\frac{z^2}{(1 - z^2)^3} \quad , \tag{32}$$

for $|z| < 1$. A proof of this formula is given in the Appendix. We also verified it numerically for a wide range of values of z.

Now, in order to compare $W^{M,em}$ with $W^{S,em}$, we shall cast R_M as a function of Ω by imposing the condition that the scalar source be in circular orbit due to the influence of a gravitational force. There is no unique way to define a gravitational field in flat spacetime (see, e.g., [19]). Assuming Newtonian gravity and using Kepler's third law $R_M(\Omega) = (M\Omega^{-2})^{1/3}$ in (30), we find $W^{M,em}$ as a function of Ω. We plot the ratio $W^{S,em}/W^{M,em}$ in figure 3 as a function of Ω. A similar result is obtained if one considers the gravitational force generated by the one-graviton exchange diagram [19]. In this case a straightforward calculation gives $\gamma^{2/3} R_M = (M\Omega^{-2})^{1/3}$ where γ is defined below equation (28). By solving this equation for R_M, we obtain

$$R_M(\Omega) = \frac{2^{\frac{1}{3}}\, M^2\, \Omega^2}{3\left(27\, M\, \Omega^4 - 2\, M^3\, \Omega^6 + \sqrt{-4\, M^6\, \Omega^{12} + (27\, M\, \Omega^4 - 2\, M^3\, \Omega^6)^2}\right)^{\frac{1}{3}}} + \frac{\left(27\, M\, \Omega^4 - 2\, M^3\, \Omega^6 + \sqrt{-4\, M^6\, \Omega^{12} + (27\, M\, \Omega^4 - 2\, M^3\, \Omega^6)^2}\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}\; 3\, \Omega^2} - \frac{M}{3}\,, \qquad (33)$$

where $R_M > 0$. We substitute (33) in (30) to find the radiated power $W^{M,em}$. We plot in figure 4 the ratio $W^{S,em}/W^{M,em}$ as a function of Ω. Asymptotically, the radiated power is

$$W^{M,em}\Big|_{R \gg 2M} \approx q^2 M^{2/3} \Omega^{8/3}/12\pi \qquad (34)$$

in either case. Figure 3 and figure 4 show that for $\Omega \to 0$ the ratio $W^{S,em}/W^{M,em}$ tends to unity as it should [see equations (24) and (34)]. As the source approaches the last stable orbit, however, our numerical and approximate analytic results show that $W^{S,em}/W^{M,em} < 1$. This is not a trivial consequence of redshift since the frequency Ω is measured at infinity in both the Schwarzschild and Minkowski calculations.

VI. ASYMPTOTIC RADIATION

Finally, it is interesting to compute the amount of the radiated power that escapes to infinity, namely

$$W^{S,obs} = \sum_{l=1}^{\infty} \sum_{m=1}^{l} \left[|\, \overrightarrow{\mathcal{T}}_{\omega_0 l}\, |^2\, \overrightarrow{W}^{S,em}_{lm} + |\, \overleftarrow{\mathcal{R}}_{\omega_0 l}\, |^2\, \overleftarrow{W}^{S,em}_{lm} \right] \quad . \qquad (35)$$

By using $|\, \overleftarrow{\mathcal{R}}_{\omega_0 l}\, |^2 = |\, \overrightarrow{\mathcal{R}}_{\omega_0 l}\, |^2 = 1 - |\, \overrightarrow{\mathcal{T}}_{\omega_0 l}\, |^2$, the observed power (35) can be written as

$$W^{S,obs} = \sum_{l=1}^{\infty} \sum_{m=1}^{l} \left[|\, \overrightarrow{\mathcal{T}}_{\omega_0 l}\, |^2 (\overrightarrow{W}^{S,em}_{lm} - \overleftarrow{W}^{S,em}_{lm}) + \overleftarrow{W}^{S,em}_{lm} \right] \quad . \qquad (36)$$

A numerical estimate of $W^{S,obs}/W^{S,em}$ is given by the solid line in figure 5. In order to obtain an analytic approximation to the transmission coefficient $|\, \overrightarrow{\mathcal{T}}_{\omega_0 l}\, |^2$ and hence to $W^{S,obs}$ (for $\omega_0^2/V_S^{\max} \ll 1$) we first note that asymptotically (19) reads

$$\psi^{\vec{S}}_{\omega_0 l}(r) \approx \frac{(2M)^{l+1}(l!)^2}{(2l+1)!r^l} \quad \text{for} \quad r \gg 2M \quad . \tag{37}$$

Now, by fitting (37) with (13) (with $\omega = \omega_0$) in the range of r satisfying $r \gg 2M$ and $\omega_0 r \ll 1$, we obtain

$$|\, \vec{\mathcal{T}}_{\omega_0 l} \,| = \frac{2^{2l+2}(l!)^3(M\omega_0)^{l+1}}{(2l+1)!(2l)!} \quad . \tag{38}$$

The analytic approximation of $W^{S,obs}/W^{S,em}$ obtained in this way is given by the dashed line in figure 5. It is seen that very little of the emitted radiation is absorbed by the black hole. This does not contradict the fact that Schwarzschild black holes have a non-negligible absorption cross section for infrared particles [20] (actually of the order of the horizon area), because the main contribution to the cross section comes from modes with $l = 0$, which are not emitted by our circularly moving source.

VII. FINAL REMARKS

In summary, we have calculated the radiated power from a scalar source rotating around a black hole in the framework of quantum field theory at the tree level. We have shown that for relativistic circular orbits the emitted power is about 30% and 20% smaller than what would be obtained in Newtonian gravity and in flat-spacetime theory with gravitation generated by one-graviton exchange, respectively. This is in agreement with the fact that astrophysical processes involving wavelengths of the order of the Schwarzschild radius need to be described using fully curved spacetime. We have also shown that most of the emitted energy escapes to infinity. Clearly, the presence of surrounding matter could trap the emitted radiation in the vicinity of the black hole because of friction effects [4]. These astrophysical issues, however, are beyond the scope of the present paper. Our conclusions are qualitatively in agreement with those obtained for gravitational waves. The procedure used here can be readily adapted for scalar sources following other trajectories (see, e.g., [21] for chaotic ones) by altering the scalar source (2).

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APPENDIX: A PROOF OF EQ. (32)

In this Appendix we prove the following formula:

$$f(z) = \sum_{l=1}^{\infty}\sum_{m=1}^{l} m^2 [j_l(mz)]^2 |Y_{lm}(\pi/2, \phi)|^2 = \frac{1}{24\pi}\frac{z^2}{(1-z^2)^3} \quad (|z| < 1) \quad . \tag{32}$$

We note here that $|Y_{lm}(\pi/2,\phi)|^2 = |Y_{lm}(\pi/2,0)|^2$ is independent of ϕ.

We start by showing that

$$f(z) = \frac{1}{32\pi^2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\, [g(z\sin\theta,\phi)]^2 \quad , \tag{39}$$

where

$$g(a,\phi) \equiv \int_{-\infty}^{+\infty} d\psi\, \delta'(\phi - \psi + a\sin\psi) \quad , \tag{40}$$

if $-1 < a < 1$. By combining the formulae

$$e^{ikz\cos\gamma} = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kz)\mathrm{P}_l(\cos\gamma) \tag{41}$$

and

$$\mathrm{P}_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}(\theta',\psi)^* Y_{lm}(\theta,\phi) \quad , \tag{42}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\psi)$, and letting $\theta' = \pi/2$, we have

$$e^{ikz\sin\theta\cos(\phi-\psi)} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l j_l(kz) Y_{lm}(\pi/2,\psi)^* Y_{lm}(\theta,\phi) \quad , \tag{43}$$

By using the formula $(2\pi)^{-1}\int_{-\infty}^{+\infty} d\psi\, e^{i\mu\psi} = \delta(\mu)$, we find

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\psi\, e^{ikz\sin\theta\cos(\phi-\psi)+ik\psi} \\ &= 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l} i^l\, j_l(kz) Y_{lm}(\pi/2,0) Y_{lm}(\theta,\phi)\delta(k-m) \quad . \end{aligned} \tag{44}$$

By multiplying by k and integrating over k we have

$$\begin{aligned} G(z,\theta,\phi) &\equiv \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\psi\, k\, e^{ikz\sin\theta\cos(\phi-\psi)+ik\psi} \\ &= \sum_{l=1}^{\infty}\sum_{m=1}^{l} i^l\, m\, j_l(mz) Y_{lm}(\pi/2,0)[Y_{lm}(\theta,\phi) - Y_{l,-m}(\theta,\phi)] \quad . \end{aligned} \tag{45}$$

(We are using the convention $Y_{l,-m}(\theta,0) = Y_{lm}(\theta,0)$ here.) Notice that this function is periodic in ϕ with priod 2π. By using orthonormality of the spherical harmonics $Y_{lm}(\theta,\phi)$ we find

$$f(z) = \frac{1}{2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\, |G(z,\theta,\phi)|^2 \quad . \tag{46}$$

By shifting the integration variable in (45) as $\psi \to \psi + \phi$, we have

$$
\begin{aligned}
G(z,\theta,\phi) &= \frac{1}{8\pi^2}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} d\psi\, k\, e^{ik(z\sin\theta\cos\psi+\psi+\phi)} \\
&= -i\frac{\partial}{\partial\phi}\left[\frac{1}{8\pi^2}\int_{-\infty}^{+\infty} d\psi \int_{-\infty}^{+\infty} dk e^{ik(z\sin\theta\cos\psi+\psi+\phi)}\right] \\
&= -\frac{i}{4\pi}\int_{-\infty}^{+\infty} d\psi\, \delta'(z\sin\theta\cos\psi+\psi+\phi) \quad .
\end{aligned}
\tag{47}
$$

By changing the variable as $\psi \to \pi/2 - \psi$ we find

$$
\begin{aligned}
G(z,\theta,\phi) &= -\frac{i}{4\pi}\int_{-\infty}^{+\infty} d\psi\, \delta'(\phi+\pi/2-\psi+z\sin\theta\sin\psi) \\
&= -\frac{i}{4\pi}\, g(z\sin\theta, \phi+\pi/2) \quad ,
\end{aligned}
\tag{48}
$$

where the function $g(a,\phi)$ is defined by (40). By substituting this in (46) and using the fact that $g(a,\phi)$ is periodic in ϕ with period 2π, we obtain (39).

Now, equation (39) can be rewritten as

$$
\begin{aligned}
f(z) &= \frac{1}{32\pi^2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\, [g(z\sin\theta,\phi)]^2 \\
&= \frac{1}{32\pi^2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta \int_{-\infty}^{+\infty} d\psi_1\, \delta'(\phi-\psi_1+z\sin\theta\sin\psi_1) \\
&\qquad\qquad \times \int_{-\infty}^{+\infty} d\psi_2\, \delta'(\phi-\psi_2+z\sin\theta\sin\psi_2) \\
&= -\frac{1}{32\pi^2}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta \int_{-\infty}^{+\infty} d\psi_1\, \delta(\phi-\psi_1+z\sin\theta\sin\psi_1) \\
&\qquad\qquad \times \int_{-\infty}^{+\infty} d\psi_2\, \delta''(\phi-\psi_2+z\sin\theta\sin\psi_2) \quad .
\end{aligned}
\tag{49}
$$

The function $\psi - z\sin\theta\sin\psi$ is a monotonously increasing function of ψ for $|z| < 1$, and the equation $\phi = \psi - z\sin\theta\sin\psi$ is satisfied for $\phi = 0$ by $\psi = 0$ and for $\phi = 2\pi$ by $\psi = 2\pi$. Hence, this equation for ψ has a solution with a value of ϕ in $[0, 2\pi]$ only if $0 \le \psi \le 2\pi$. Hence, after performing the ϕ integral, the integration range for ψ_1 becomes $[0, 2\pi]$. Thus, we obtain

$$
f(z) = -\frac{1}{32\pi^2}\int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\psi_1 \int_{-\infty}^{+\infty} d\psi_2\, \delta''(\chi_1-\chi_2) \quad ,
\tag{50}
$$

where $\chi_i = \psi_i - z\sin\theta\sin\psi_i$, $(i = 1, 2)$. Hence,

$$
\begin{aligned}
f(z) &= -\frac{1}{32\pi^2}\int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\chi_1 \frac{d\psi_1}{d\chi_1} \int_{-\infty}^{+\infty} d\chi_2 \frac{d\psi_2}{d\chi_2}\, \delta''(\chi_1-\chi_2) \\
&= \frac{1}{32\pi^2}\int_0^{\pi} d\theta\, \sin\theta \int_0^{2\pi} d\psi \left[\frac{d^2\psi}{d\chi^2}\right]^2 \frac{d\chi}{d\psi} \\
&= \frac{z^2}{32\pi^2}\int_0^{\pi} d\theta \sin^3\theta \int_0^{2\pi} d\psi\, \frac{\sin^2\psi}{(1-z\sin\theta\cos\psi)^5} \quad ,
\end{aligned}
\tag{51}
$$

where we have defined $\chi = \psi - z\sin\theta\sin\psi$. By integration by parts with respect to ψ, we obtain

$$f(z) = \frac{z}{128\pi^2} h(z) \quad , \tag{52}$$

where

$$h(z) = \int_0^\pi d\theta \, \sin^2\theta \int_0^{2\pi} d\psi \frac{\cos\psi}{(1 - z\sin\theta\cos\psi)^4} \quad . \tag{53}$$

Now we evaluate this integral. Define

$$H(z) \equiv \frac{1}{3}\int_0^\pi d\theta \, \sin\theta \int_0^{2\pi} \frac{d\psi}{(1 - z\sin\theta\cos\psi)^3} \quad . \tag{54}$$

The ψ integral can be performed by using the residue theorem after the substitution $t = e^{i\psi}$. The result is

$$H(z) = \frac{2\pi}{3}\int_0^\pi d\theta \, \sin\theta \left[\frac{1}{(1 - z^2\sin^2\theta)^{3/2}} + \frac{3}{2}\frac{z^2\sin^2\theta}{(1 - z^2\sin^2\theta)^{5/2}}\right] \quad . \tag{55}$$

Then, by using the formulae

$$\int \frac{dx}{(a + cx^2)^{3/2}} = \frac{1}{a}\frac{x}{\sqrt{a + cx^2}} \quad , \tag{56}$$

$$\int \frac{dx}{(a + cx^2)^{5/2}} = \frac{1}{a^2}\left[\frac{x}{\sqrt{a + cx^2}} - \frac{cx^3}{3(a + cx^2)^{3/2}}\right] \tag{57}$$

with $a = 1 - z^2$ and $c = z^2$ in (55) after the substitution $x = \cos\theta$, we have

$$H(z) = \frac{4\pi}{3}\frac{1}{(1 - z^2)^2} \quad . \tag{58}$$

Noting that $h(z) = H'(z)$, we find from this

$$h(z) = \frac{16\pi}{3}\frac{z}{(1 - z^2)^3} \quad . \tag{59}$$

By substituting this result in (52) we obtain equation (32).

REFERENCES

[1] van Paradijs J and McClintock J E 1995 *X-Ray Binaries*, eds. Lewin W H G, van Paradijs J and van den Heuvel E P J (Cambridge University Press, Cambridge)

[2] Rees M J 1998 *Black holes and relativistic stars*, ed. Wald R M (The University of Chicago Press, Chicago)

[3] Shakura N I and Sunyaev R A 1973 *Astr. and Astrophys.* **24** 337
Novikov I D and Thorne K S 1973 *Black holes*, eds. DeWitt C and DeWitt B (Gordon and Breach, New York)
Lynden-Bell D and Pringle J E 1974 *Mon. Not. R. Astr. Soc.* **168** 603
Page D and Thorne K 1974 *Ap. J.* **191** 499
Frank J, King A and Raine D 1992 *Accretion Power in Astrophysics* (Cambridge University Press, Cambridge)

[4] Narayan R and Yi I 1994 *Ap. J. Lett.* **428** L13 ; *Ap. J.* 1995 **444** 231 ; *Ap. J.* 1995 **452** 710
Abramowicz M, Chen X, Kato S, Lasota J P and Regev O 1995 *Ap. J. Lett.* **438** L37
Rees M J, Begelman M C, Blandford R D and Phinney E S 1982 *Nature* **295** 17
Narayan R, Yi I and Mahadevan R 1995 *Nature* **374** 623
Narayan R, McClintock J E and Yi I 1996 *Ap. J.* **457** 821
Mahadevan R 1997 *Ap. J.* **477** 585

[5] Thorne K S 1995 *Ann N.Y. Acad. Science* **759** 127
Kennefick D 1998 *Phys. Rev. D* **58** 064012

[6] Tanaka Y *et al* 1995 *Nature* **375** 659
Bromley B C, Miller W A and Pariev V I 1998 *Nature* **391** 54

[7] Higuchi A, Matsas G E A and Sudarsky D 1997 *Phys. Rev. D* **56** R6071

[8] Higuchi A, Matsas G E A and Sudarsky D 1998 *Phys. Rev. D* **58** 104021

[9] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (Dover Publications, New York)

[10] Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (McGraw-Hill, New York)

[11] Hartle J B and Hawking S W 1976 *Phys. Rev. D* **13** 2188
Unruh W G 1976 *Phys. Rev. D* **14** 870
Israel W 1976 *Phys. Lett. A* **57** 107
Kay B S and Wald R M 1991 *Phys. Rep.* **207** 49

[12] Price R H 1972 *Phys. Rev. D* **5** 2419
Jensen B P and Candelas P 1986 *Phys. Rev. D* **33** 1590 ; Erratum 1987 **35** 4041

[13] Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series, and Products* (Academic, New York)

[14] Wald R M 1984 *General Relativity* (The University of Chicago Press, Chicago)

[15] In [8], equation (7.3) and the one just above equation (7.5) should read $\psi_{0l}(x) \approx C_2\,(2l)!\,x^{l+1}/(l!)^2$ and $C_2 = (2^{2l+1}(l!)^3 M^{l+1}\,\omega^l)/((2l)!\,(2l+1)!)$ respectively. This does not affect other results in [8].

[16] Higuchi A and Matsas G E A 1993 *Phys. Rev. D* **48** 689

[17] Poisson E 1993 *Phys. Rev. D* **47** 1497 ; 1995 *Phys. Rev. D* **52** 5719
Cutler C, Finn L S, Poisson, E and Sussman G J 1993 *Phys. Rev. D* **47** 1511
Tagoshi H and Nakamura T 1994 *Phys. Rev. D* **49** 4016
Tagoshi H and Sasaki M 1994 *Prog. Theor. Phys.* **92** 745

Poisson E and Sasaki M 1995 *Phys. Rev. D* **51** 5753
Tanaka T, Tagoshi H and Sasaki M 1996 *Prog. Theor. Phys.* **96** 1087
Leonard S W and Poisson E 1997 *Phys. Rev. D* **56** 4789

[18] Jackson J D 1975 *Classical Electrodynamics*, 2nd edition (John Wiley & Sons, New York)
Barut A O 1980 *Electrodynamics and Classical Theory of Fields and Particles* (Dover Publications, New York)

[19] Misner C W, Thorne K S and Wheeler J A 1971 *Gravitation* (Freeman, San Francisco)

[20] Page D 1976 *Phys. Rev. D* **13** 198
Unruh W G 1976 *Phys. Rev. D* **14** 3251
Das S R, Gibbons G and Mathur S D 1997 *Phys. Rev. Lett.* **78** 417

[21] Vieira W M and Letelier P S 1996 *Phys. Rev. Lett.* **76** 4098
Letelier P S and Vieira W M 1997 *Phys. Rev. D* **56**, 8095

FIGURES

FIGURE 1

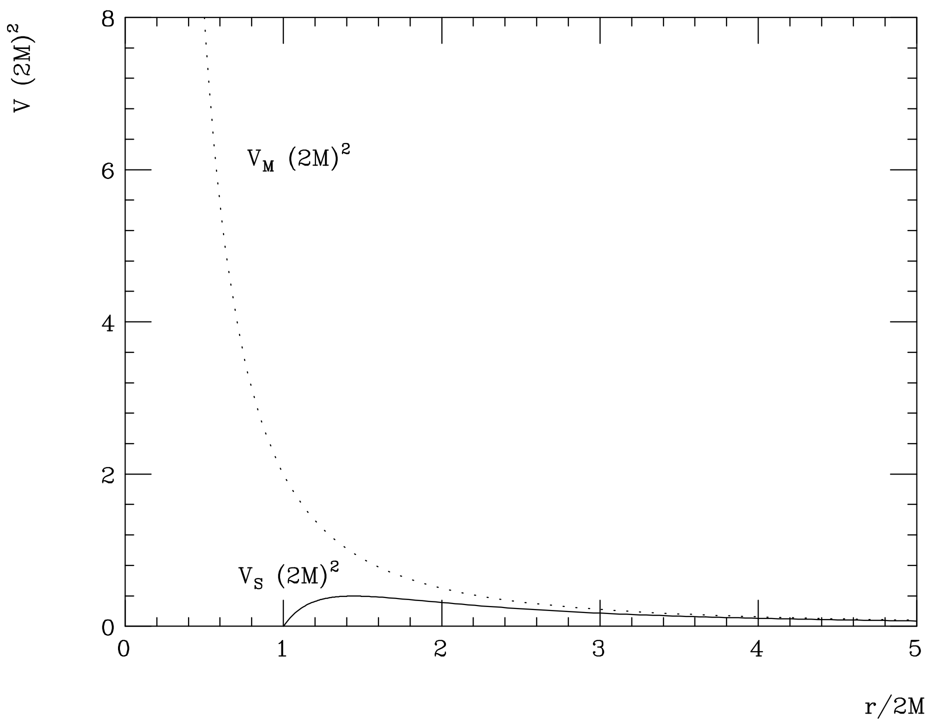

FIG. 1. Scattering potentials V_M and V_S are plotted as functions of $r/2M$ for $l = 1$, where we recall that V_M and V_S are defined as functions of Minkowski and Schwarzschild r coordinates, respectively. Asymptotically V_M and V_S fall as $1/r^2$. V_S is only defined outside the black hole ($r > 2M$). Because of the nonexistence of the event horizon in Minkowski spacetime, V_M is also defined in the region $0 < r \leq 2M$.

FIGURE 2

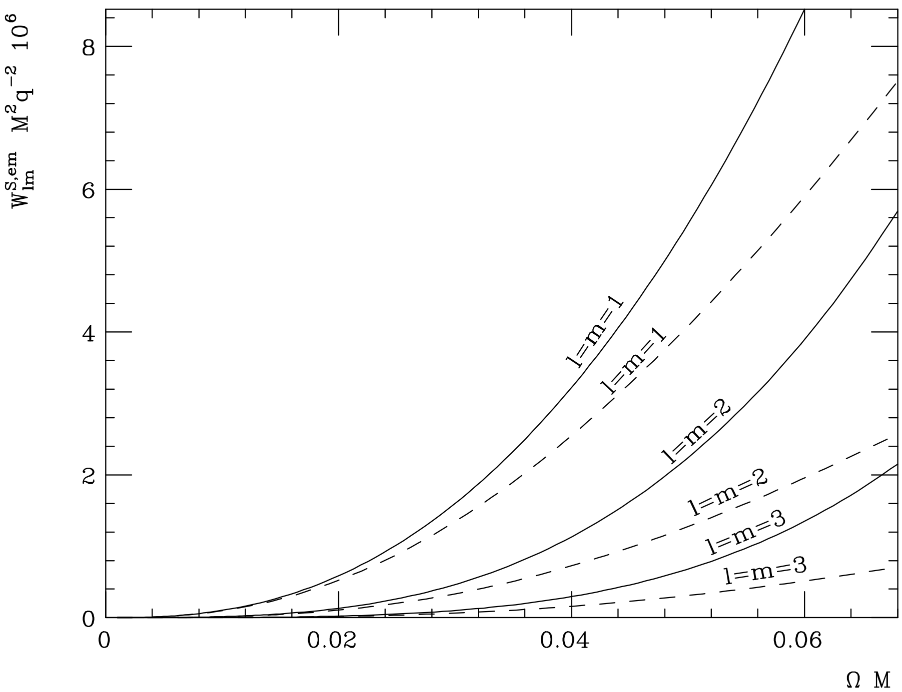

FIG. 2. The radiated power $W_{lm}^{S,em}$ is plotted as a function of Ω for geodesic orbits. Solid and dashed lines are associated with the numerical calculation and analytic approximation, respectively. The maximum ΩM considered is 0.068 which is associated with the last stable circular orbit. As expected our analytic approximation is only accurate to describe the emission of low-energy particles as it can be seen from the coincidence of the numerical and (approximate) analytic curves for small Ω. It is clear that small angular-momentum waves give the main contribution to the total radiated power.

FIGURE 3

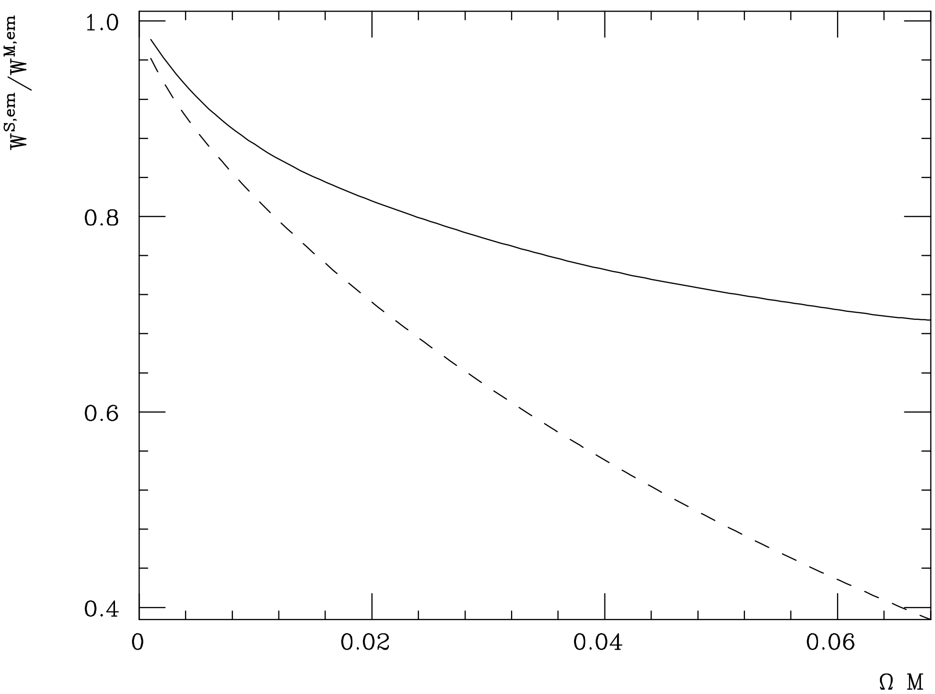

FIG. 3. The $W^{S,em}/W^{M,em}$ ratio is plotted as a function of Ω as measured by asymptotic observers, where the summations involved in the calculation of $W^{S,em}$ and $W^{M,em}$ were performed up to $l = 3$. $W^{S,em}$ and $W^{M,em}$ are the power emitted by an orbiting source as calculated by asymptotic observers who assume Schwarzschild spacetime and Minkowski spacetime with Newtonian gravity respectively. The graph is plotted up to $\Omega M = 0.068$, since this is the faster stable circular orbit according to General Relativity. Solid and dashed lines are associated with the numerical calculation and analytic approximation, respectively. Asymptotically ($\Omega \to 0$) we have $W^{S,em}/W^{M,em} \to 1$. As Ω increases, however, we see that $W^{S,em}/W^{M,em}$ decreases.

FIGURE 4

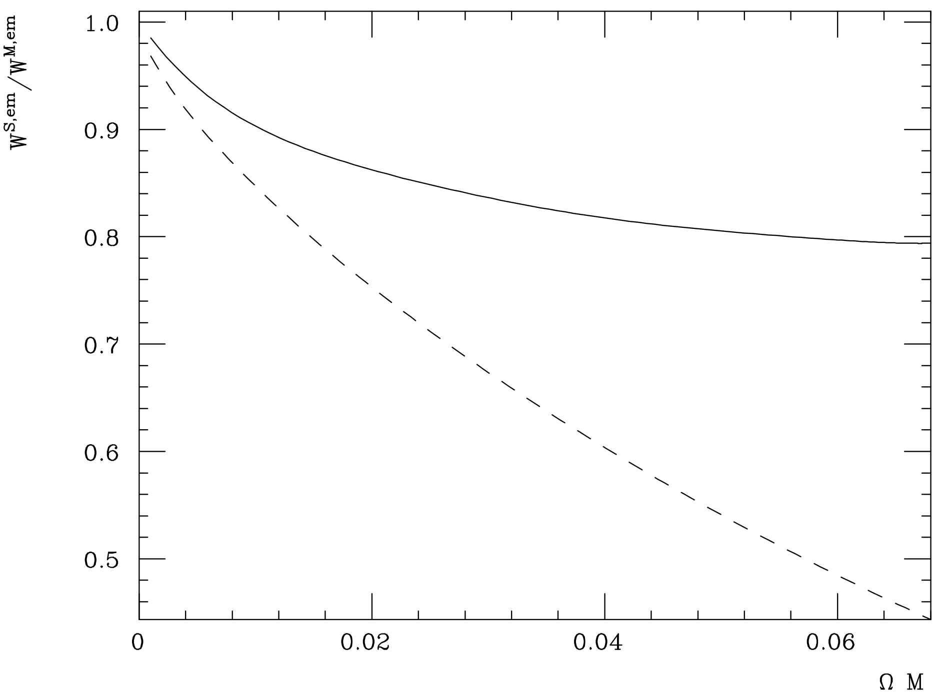

FIG. 4. Similarly to figure 3, the $W^{S,em}/W^{M,em}$ ratio is plotted as a function of Ω. Here, however, $W^{M,em}$ is the power emitted by an orbiting source as calculated by asymptotic observers who assume Minkowski spacetime with a gravitational force induced by gravitons, rather than with the usual Newtonian force. We note that figure 3 and figure 4 have similar features: (i) $W^{S,em}/W^{M,em} \to 1$ asymptotically ($\Omega \to 0$) and, (ii) $W^{S,em}/W^{M,em}$ decreases up to from 20% (see figure 4) to 30% (see figure 3) as Ω increases.

FIGURE 5

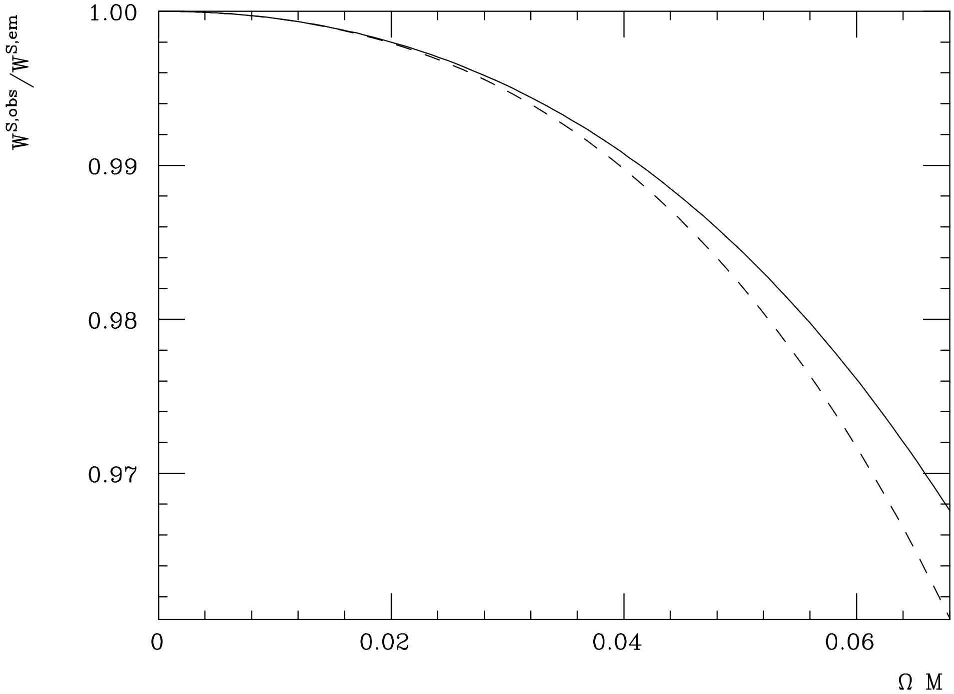

FIG. 5. The ratio $W^{S,obs}/W^{S,em}$ is plotted as a function of Ω for geodesic orbits, where the summations involved in the calculation of $W^{S,obs}$ and $W^{S,em}$ were performed up to $l = 3$. Solid and dashed lines are associated with the numerical calculation and analytic approximation, respectively. We note that most of the emitted energy is radiated away to infinity.